List Decoding of Algebraic Codes

Peter Beelen, Kristian Brander and Johan S.R. Nielsen

DTU Mathematics
Technical University of Denmark

Contents

1. List decoding of error-correcting codes
2. Fast list decoding of Reed–Solomon codes
3. Fast list decoding of certain AG codes
4. Wu decoding of Reed–Solomon codes

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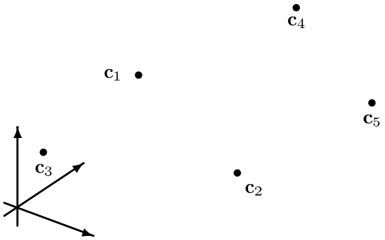
Codewords and unique decoding



- Codewords: Vectors $\mathbf{c} \in \Sigma^n$. Code: $\mathcal{C} = \{\mathbf{c}_1, \ldots, \mathbf{c}_m\}$.

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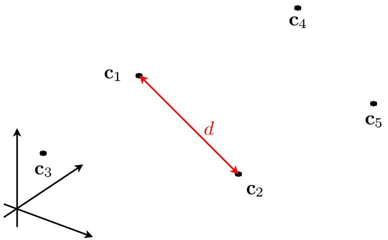

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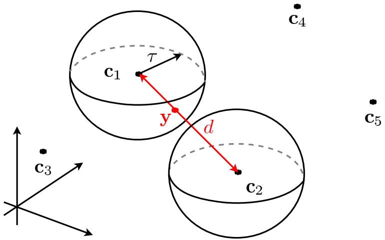

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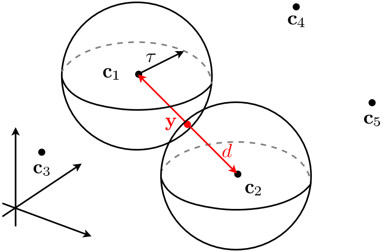
- Minimum distance, d, is the minimal number of disagreeing positions between any two codewords.
- If the number of errors, τ, is less than $\frac{d}{2}$ then there is at most one codeword within distance τ from any received word $\mathbf{y}$.

List decoding



- If $\tau \geq \frac{d}{2}$ there might be a "small" list of codewords within distance τ from $\mathbf{y}$.
- The decoder thus get a list of candidate messages.

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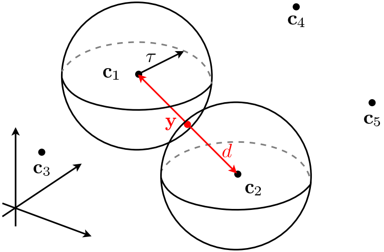

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- We require the lists to be polynomially bounded in the code length n.

Error-correcting codes and list decoding

- The rate of an error-correcting code is rate $R = \frac{\log_{|\Sigma|}(|\mathcal{C}|)}{n}$.

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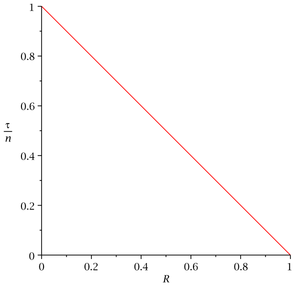

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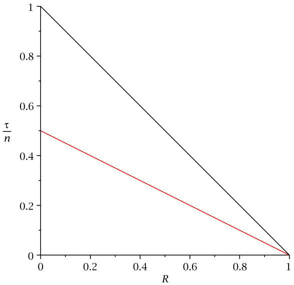

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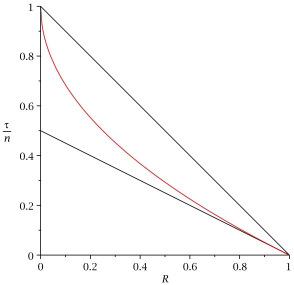

- Unique decoding: $\tau/n < \frac{1}{2}(1-R)$.
- Guruswami–Sudan algorithm: $\tau/n < 1-\sqrt{R}$.

- Furthermore: The code must be efficiently list decodable.

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Reed–Solomon codes

- A Reed–Solomon code of length n and rate $R = k/n$:

$$\mathcal{C} = \{(f(\alpha_1), \dots, f(\alpha_n)) \mid f(x) \in \mathbb{F}_q[x], \deg(f) < k\},$$

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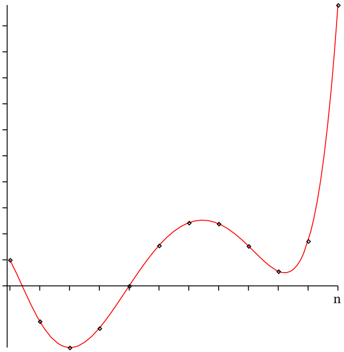

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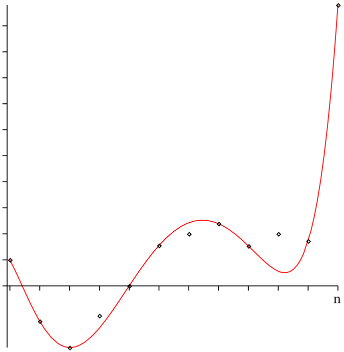
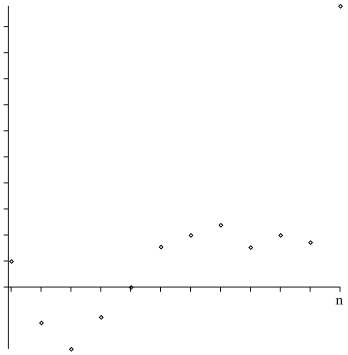


List decoding Reed–Solomon codes

- A list decoder must find $f(x) \in \mathbb{F}_q[x]$, with $\deg(f) < k$, that passes through $n - \tau$ of the received points.



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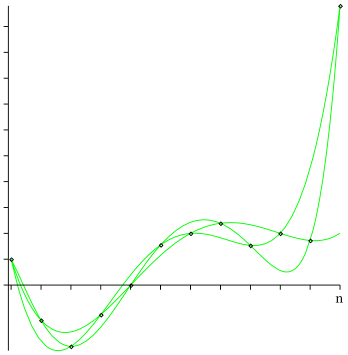

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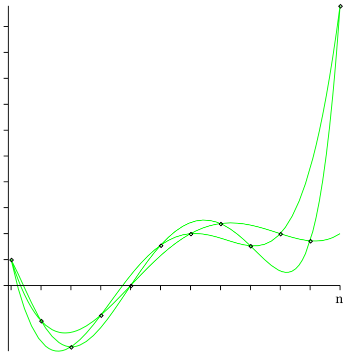

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- ... of least weighted degree.

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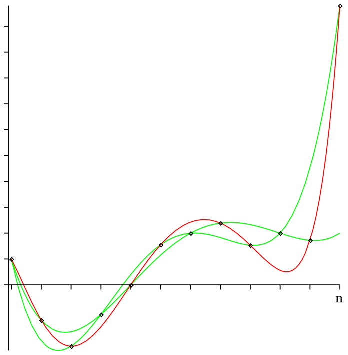

- Interpolate $Q(x, y)$ through received points, with multiplicity s.
- ... of least weighted degree.

$$\deg_w(x^i y^j) = i + (k-1)j$$

- If $\tau/n < 1 - \sqrt{R}$ then

$$Q(x, f(x)) = 0$$

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- The $\mathbb{F}_q[x]$–module of interpolation polynomials with $\deg_y(Q) \le \ell$, is spanned by

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where $E(x) = \prod_{i=1}^n (x - \alpha_i)$ and $R(\alpha_i) = y_i$ for $1 \le i \le n$.

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- Then,

$$Q(x,y) = \sum_{i=0}^{\ell} q_i(x) y^i \in \mathbb{F}_q[x,y],$$

is an interpolation polynomial if and only if $\mathbf{q} = (q_0, \ldots, q_\ell)$ is in the $\mathbb{F}_q[x]$–column span of $\mathbf{A}$.

Interpolation

- For $s = 2$ and $\ell = 3$,

$$\mathbf{A} = \begin{bmatrix} E^2 & -ER & R^2 & -R^3 \\ 0 & E & -2R & 3R^2 \\ 0 & 0 & 1 & -3R \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

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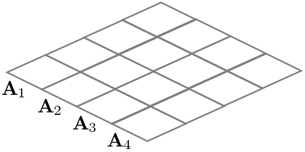
- The column span of **A** gives all interpolation polynomials. We look for short vectors, with respect to weighted degree.
- Gaussian elimination-style algorithm: Cancel highest terms.

Algorithm: Gaussian elimination

- Represent matrix as grid.



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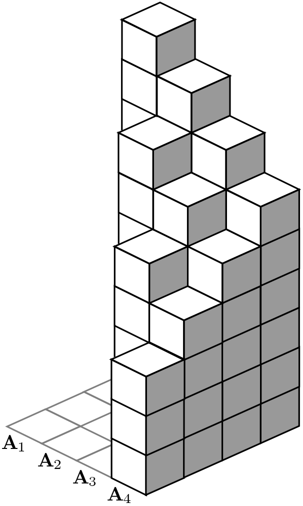

- Represent matrix as grid.
- Represent (i, j)-th entry by stack of cubes:

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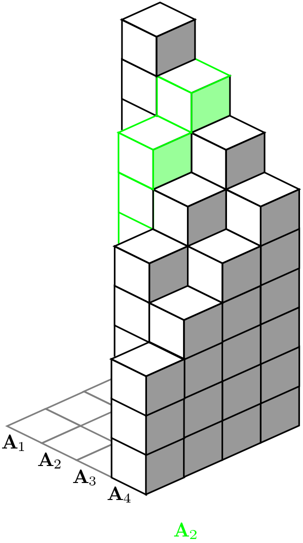

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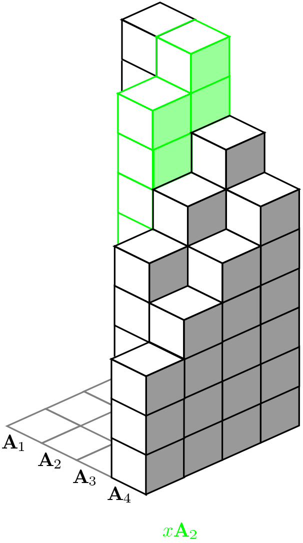

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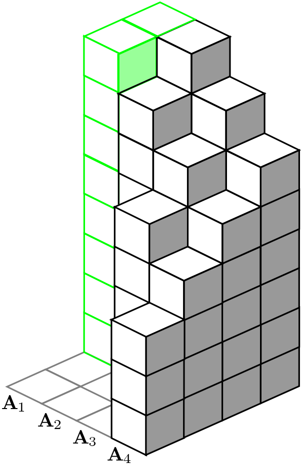

$\mathbf{A}_1 + \alpha x \mathbf{A}_2 \to \mathbf{A}_1$

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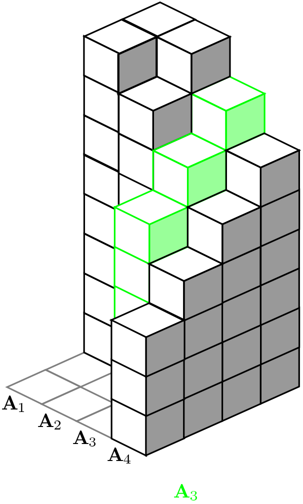

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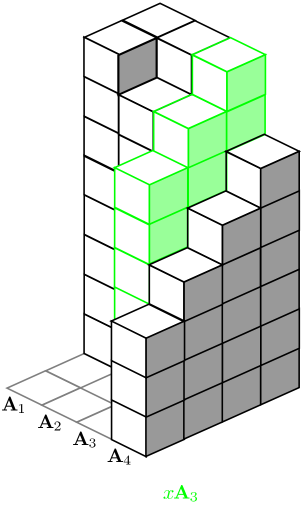

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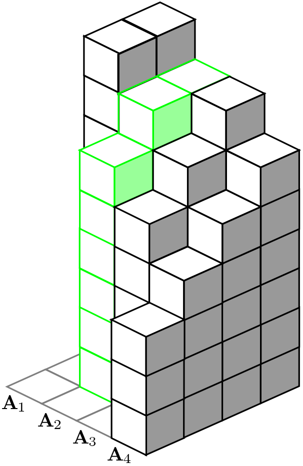

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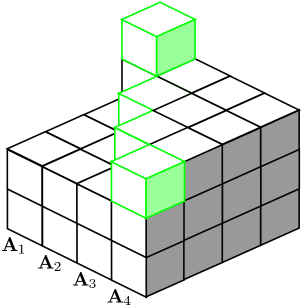

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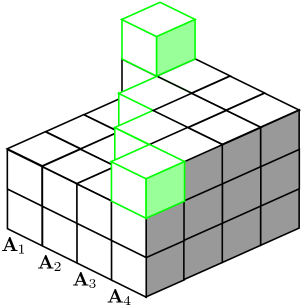

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- Continue the process, until leading coordinates occur in distinct rows.
- Leads to algorithm requiring $\mathcal{O}\left(\ell^5 n^2\right)$ $\mathbb{F}_q$-multiplications.

Algorithm: Divide and conquer

- Extend and generalize idea behind divide and conquer algorithm by Alekhnovich.
- Introduce matrix $\mathbf{U}(\mathbf{A}, t)$ representing the column operations made when "cutting down" the stack, i.e.
  - $\deg_w(\mathbf{A} \cdot \mathbf{U}(\mathbf{A}, t)) \leq \deg_w(\mathbf{A}) - t$ or
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$$\mathbf{U}(\mathbf{A}, t) = \mathbf{U}(\mathbf{A}, \lceil t/2 \rceil) \cdot \mathbf{U}(\mathbf{A}', t - d),$$

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- Leads to divide and conquer algorithm. Handle base case $\mathbf{U}(\mathbf{A}, 1)$ by Gaussian elimination.

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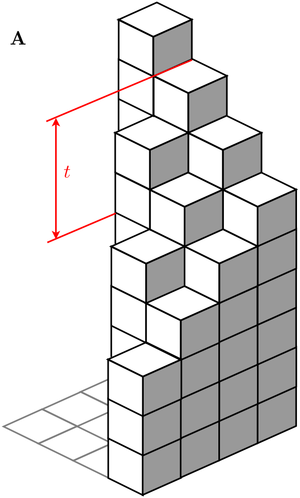

- Subproblems are easy:

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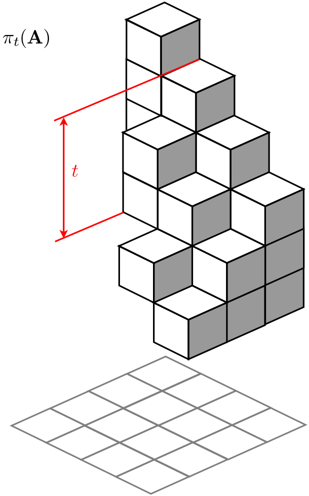

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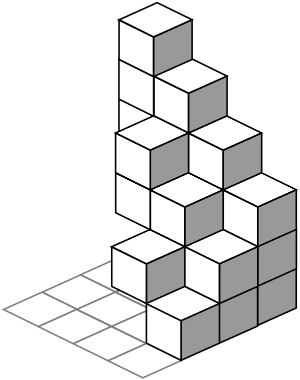

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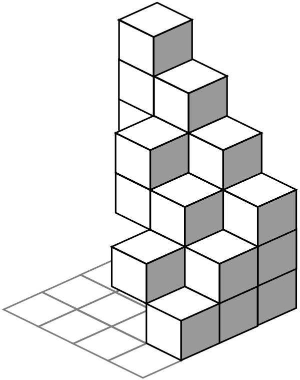

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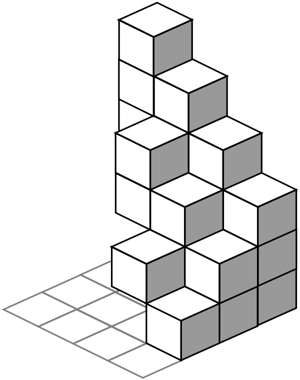

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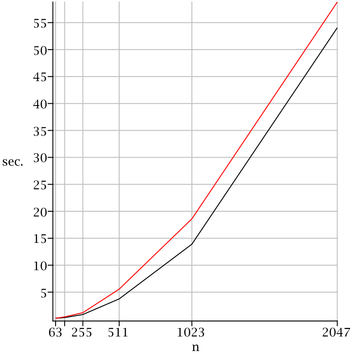
- Leads to algorithm requiring

$$\mathcal{O}\left(\ell^5 n \log^2(\ell n) \log\log(\ell n)\right)$$

$\mathbb{F}_q$-multiplications.

Comparison and conclusions

- The divide and conquer algorithm is asymptotically faster than Gaussian elimination.



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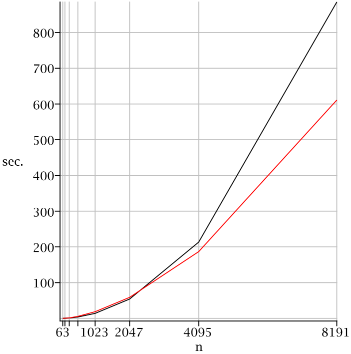

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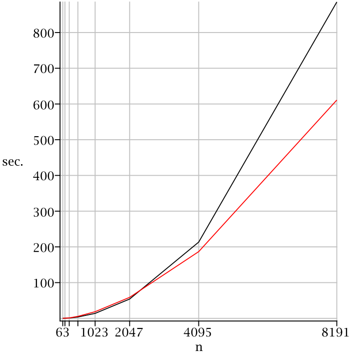

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- The algorithm works in a more general setting: list decoding of certain algebraic geometry codes.

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AG codes

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- An AG code from a simple C_{ab}-curve of length n:

$$\mathcal{C} = \{(f(\alpha_1), \ldots, f(\alpha_n)) \mid f(x) \in L(\mu P_\infty), v_{P_\infty}(f) + \mu \geq 0\},$$

Alphabet is $\Sigma = \mathbb{F}_q$ and $\alpha_1, \ldots, \alpha_n \in \mathfrak{C}(\mathbb{F}_q)$ are distinct affine points.

List decoding AG codes

- A list decoder must find $f(x) \in \mathbb{F}_q[x_1, x_2]/(F(x_1, x_2))$, with $v_{P_\infty}(f) + \mu \geq 0$, that passes through $n - \tau$ of the received points.

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- If $\tau/n < 1 - \sqrt{R}$ then

$$Q(x_1, x_2, f(x_1, x_2)) = 0$$

Translation of the interpolation problem

- The $\mathbb{F}_q[x_1, x_2]/(F(x_1, x_2))$–module of interpolation polynomials with $\deg_y(Q) \leq \ell$, is spanned by

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- Find a generating set of the module viewed as $\mathbb{F}_q[x_1]$ module. One finds a generating set of cardinality $\gamma(\ell+1)$.
- Introduce matrix $\gamma(\ell+1) \times \gamma(\ell+1)$ matrix $\mathbf{A}$,

$$[\mathbf{A}]_{(ij),(i'j')} = \text{Coefficient to } x_2^i y^j \text{ in } (i', j')\text{-th basis function}$$

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- For the well-known Hermitian curve one can list-decode one-point AG codes in
$$\mathcal{O}\left(\ell^5 n^2\log^2(\ell n)\log\log(\ell n)\right)$$
$\mathbb{F}_{q^2}$-multiplications. Note that in this case $\gamma = q$, $\delta = q+1$ and $n = q^3$.

Contents

1. List decoding of error-correcting codes
2. Fast list decoding of Reed–Solomon codes
3. Fast list decoding of certain AG codes
4. Wu decoding of Reed–Solomon codes

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- Solving the key equation: use EEA on $S(x)$ and x^{n-k}. Finds $\Lambda(x)$ and $\Omega(x)$ if $2\tau < n - k + 1$.

The Wu list decoder.

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- $f_1(x)$ and $f_2(x)$ are unknown polynomials, but upper bounds on their degrees are known.

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- Work in progress: apply Wu's list decoder to other classes of codes.

Thank you for your attention!